Euler's product formula and the Ramanujan sum

Institute Colloquium, IIT Mandi (May 05, 2025)

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Abstract: We will discuss the product formula of Euler, namely,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(\frac{1}{1 - \frac{1}{p^s}} \right), \; s > 1,$$

and the mysterious claim of Ramanujan below

$$1 + 2 + 3 + \cdots = -\frac{1}{12}.$$

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Roots of polynomial and trigonometric functions

polynomials of degree 2

If a and b are non-zero real numbers then they are roots of the equation

$$(1-\frac{x}{a})(1-\frac{x}{b})=0$$

which is the same as the equation

$$1-(\frac{1}{a}+\frac{1}{b})x+\frac{1}{ab}x^2=0$$

Thus the negative of the coefficient of x is the sum of the reciprocal of the roots.

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Thus the negative of the coefficient of x is the sum of the reciprocal of the roots. Replace x by x^2 and a, b by a^2, b^2:

$$(1-\frac{x^2}{a^2})(1-\frac{x^2}{b^2})=1-(\frac{1}{a^2}+\frac{1}{b^2})x^2+\frac{1}{a^2b^2}x^4=0$$

Now the roots are $\pm a$, $\pm b$.

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Now the roots are $\pm a$, $\pm b$. Thus the coefficient of x^2 is the sum of the squares of the reciprocals of the positive roots.

degree $3, 4 \ldots$

Clearly, if we pick any three non-zero real numbers a, b, c then $\pm a, \pm b, \pm c$ are the roots of the equation

$$(1 - \frac{x^2}{a^2})(1 - \frac{x^2}{b^2})(1 - \frac{x^2}{c^2}) = 0$$

or the equation

$$1 - (\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})x^2 + (\frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2a^2})x^4 - \frac{1}{a^2b^2c^2}x^6 = 0.$$

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Again, we see that *the coefficient of* x^2 *is the sum of the squares of the reciprocals of the positive roots.*

induction!

What if we pick n non-zero real numbers a_i, $i = 1, 2, \ldots, n$. Well, we can find the polynomial equation which has $\pm a_i$, $i = 1, 2, \ldots, n$ as its roots, namely

$$\prod_{i=1}^{n}(1 - \frac{x^2}{a_i^2}) = 0.$$

One final time, we rewrite this equation as

$$1 - (\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2})x^2 + \cdots = 0$$

and note: the coefficient of x^2 is the sum of the squares of the reciprocals of the positive roots.

Roots of trigonometric equations?

What about the equation

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = 0??$$

We know that the roots of the equation $\sin x = 0$ are

$$0, \pm\pi, \pm 2\pi, \ldots.$$

To ensure that the roots are non-zero, we consider the equation

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = 0$$

instead! Now, the roots are $\pm\pi, \pm 2\pi, \ldots.$

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Does it then follow that $\frac{\sin x}{x}$ admits a factorization? What sort of factorization?

Wishful thinking?

We have pointed out at the outset that if p is a polynomial with roots $\pm a_i$, $i = 1, 2, \ldots, n$ then it has the factorization: $\prod_{i=1}^{n}(1 - \frac{x^2}{a_i^2})$.

The trigonometric function $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$ is sort of like a polynomial of infinite degree – consequently, has infinitely many zeros! Thus it is natural to expect a factorization of the form:

$$\frac{\sin x}{x} = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots = \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2})$$

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Note that expanding the finite product $\prod_{n=1}^{k} \left(1 - \frac{x^2}{n^2\pi^2}\right)$, the coeffcient of x^2 is obtained by picking 1 in each of the factors except one, where we must pick the coefficient of x^2. Having accounted for all such possibilities, the coefficient of x^2 in the product is seen to be

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots \frac{1}{k^2\pi^2}.$$

Taking limit as $k \to \infty$, we see that the coefficient of x^2 in the product formula for $\frac{\sin x}{x}$ is equal to $\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots$.

The amazing formula of Euler

If $\frac{\sin x}{x}$ were to admit a factorization involving its roots, then it would be natural to expect that sum of the squares of the reciprocal of these roots equal the coefficient of x^2 in the series expansion as in the case of ordinary polynomials, that is, we would have

$$\frac{1}{3!} = \frac{1}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \cdots \frac{1}{n^2} + \cdots),$$

or equivalently,

$$\frac{\pi^2}{3!} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots \frac{1}{n^2} + \cdots$$

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This formula was discovered by Euler in 1735. He did not stop there instead he defined the function ζ on $(1, \infty)$ and discovered all the values of ζ at $2k$, $k \in \mathbb{N}$ starting with $\zeta(2) = \frac{\pi^2}{6}$.

Infinitude of primes

the product formula of Euler

For $s \in \mathbb{R}$, $s > 1$, it is easy to check that the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + \dots$$

is convergent by the usual integral test. Euler proved the product formula

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all primes. His proof, in a manner of speaking, is quite straightforward.

the product formula contd.

Subtracting all the powers of 2 from $\zeta(s)$, we have

$$\begin{aligned} \zeta(s) &= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + \dots \\ -2^{-s}\zeta(s) &= \quad - 2^{-s} \quad - 4^{-s} \quad - 6^{-s} \quad - 8^{-s} \quad - \dots \end{aligned}$$

Thus, we have

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1^{-s} \quad + 3^{-s} \quad + 5^{-s} \quad + 7^{-s} \quad + 9^{-s} + \dots$$

Hence, none of the powers of $\frac{1}{2}$ survive in $(1 - \frac{1}{2^s})\zeta(s)$.

the product formula contd.

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Multiplying $(1 - \frac{1}{2^s})\zeta(s)$ with $\frac{1}{3^{-s}}$ and subtracting from $(1 - \frac{1}{2^s})\zeta(s)$, we obtain as before

$$(1 - \frac{1}{2^s})\zeta(s) - \frac{1}{3^s}(1 - \frac{1}{2^s})\zeta(s) = 1^{-s} \quad + 5^{-s} + 7^{-s} \quad + 11^{-s} + 13^{-s} + \dots.$$

and conclude that $(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s)$ has no powers of either $\frac{1}{2}$ or $\frac{1}{3}$ in it.

the product formula contd.

Repeating this process with each of the primes p, we have

$$\left(\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\cdots\right)\zeta(s) = 1,$$

or equivalently,

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}, \quad s > 1.$$

Clearly, $\zeta(s)$ as $s \to 1$ diverges to infinity. Therefore, the product representing it must also diverge to infinity proving that there must be infinitely many primes.

Leonhard Euler

More zeta values

Euler begins by examining the identity

$$\sin(x) = x \cdot \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

by taking log on both sides. Thus, he gets

$$\log(\sin(x)) = \log(x) + \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

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It is time to differentiate and get rid of the logarithms:

$$\begin{aligned} x\cot(x) &= 1 + \sum_{n=1}^{\infty} \frac{-2x^2}{n^2\pi^2 - x^2} = 1 + \sum_{n=1}^{\infty} \frac{-2x^2}{n^2\pi^2}\left(\frac{1}{1 - \frac{x^2}{n^2\pi^2}}\right) \\ &= 1 + \sum_{n=1}^{\infty} \frac{-2x^2}{n^2\pi^2}\left(1 + \frac{x^2}{n^2\pi^2} + \frac{x^4}{n^4\pi^4} + \dots\right) \\ &= 1 + \frac{-2x^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{-2x^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{-2x^6}{\pi^6}\sum_{n=1}^{\infty}\frac{1}{n^6} + \dots \end{aligned}$$

Zeta values at even numbers

Therefore, Euler proved that

$$x\cot(x) = 1 + \frac{-2x^2}{\pi^2}\zeta(2) \quad + \frac{-2x^4}{\pi^4}\zeta(4) \quad + \frac{-2x^6}{\pi^6}\zeta(6) \quad + \dots.$$

Thus, he obtained an explicit formula for the Maclaurin series of $x\cot(x)$ in terms of $\zeta(2k)$, $k = 1, 2, \dots$. On the other hand, we know that $x\cot(x)$ is of the form

$$x\cot(x) = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \cdots .$$

Zeta values at even numbers

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Therefore, Euler found the zeta values at all the even integers, namely,

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(4) = \frac{\pi^4}{90} \quad \zeta(6) = \frac{\pi^6}{945}, \dots.$$

But what about

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots = ?$$

What is $\sum_{p\in\mathbb{P}} \frac{1}{p}$?

the size of the set of primes

$$\sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1) = \infty$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Is the sum $\sum_{p \in \mathbb{P}} \frac{1}{p}$ finite or infinite?

the sum $\sum_{p \in \mathbb{P}} \frac{1}{p}$ infinite

We have already seen that

$$\prod_p \frac{1}{1 - \frac{1}{p}} = \zeta(1) = \infty.$$

Taking log on both sides, we convert the infinite product into an infinite sum:

$$-\sum_p \log\left(1 - \frac{1}{p}\right) = \infty.$$

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Now, using the series expansion of log, we have that

$$\begin{aligned} -\sum_p \log\left(1-\frac{1}{p}\right) &= -\sum_p \Big(-p^{-1} - \frac{p^{-2}}{2} - \frac{p^{-3}}{3} - \dots\Big) \\ &= \sum_p \Big(\frac{1}{p} + \sum_{k=2}^{\infty} \frac{p^{-k}}{k}\Big) \\ &= \sum_p p^{-1} + \sum_{k=2}^{\infty} \frac{p^{-k}}{k} \end{aligned}$$

Evaluating one of the two sums

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{p^{-k}}{k} &= \frac{p^{-2}}{2}+\frac{p^{-3}}{3}+\frac{p^{-4}}{4}+\dots\\ &\leq p^{-2}+p^{-3}+p^{-4}+\dots\\ &\leq \frac{p^{-2}}{1-\frac{1}{p}}\\ &\leq \frac{1}{p^2-\frac{p^2}{2}}=\frac{2}{p^2}.\end{aligned}$$

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Consequently, we have

$$\sum_p\Big(\sum_{k=2}^{\infty}\frac{p^{-k}}{k}\Big) \le \sum_p \frac{2}{p^2} \le \sum_{n=1}^{\infty}\frac{2}{n^2} = 2\zeta(2) = \frac{\pi^2}{3}.$$

It now follows that $\sum_p p^{-1}$ must be infinite. This says that there are infinitely many primes but it actually says lot more!

Infinitude of primes, second cut

Taking log on both sides of the product formula of Euler, we get

$$\log(\zeta(s)) = \sum_p -\log\left(1 - p^{-s}\right),$$

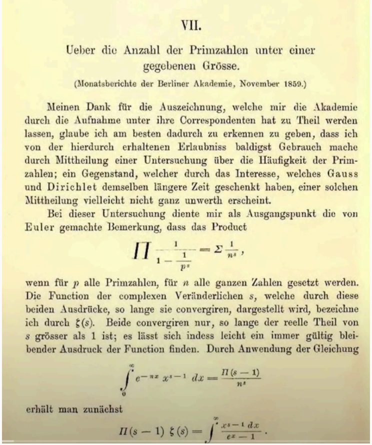
and as before, differentiating this equality, we get

$$\begin{aligned}
\frac{-\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{p^{-s}\log(p)}{1-p^{-s}} \\
&= -\sum_p \log(p)\Big(\frac{1}{1-p^{-s}} - 1\Big) \\
&= \sum_{p^k} \frac{\log(p)}{p^{ks}} \\
&= \frac{\log(2)}{2^s} + \frac{\log(3)}{3^s} + \frac{\log(2)}{4^s} + \frac{\log(5)}{5^s} + \frac{\log(7)}{7^s} + \frac{\log(2)}{8^s} + \frac{\log(3)}{9^s} + \ldots
\end{aligned}$$

Just like the product formula for the $x\sin(x)$, may be, there is a chance that $\zeta(s)$ can be expressed as a product of its zeros.

Riemann's paper

Bernhard Riemann



VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\, \zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

First page of Riemann's only paper on the zeta function

The zeta function extended to $\mathbb{H}_1$

- The Riemann zeta function defined by the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

converges only on the region

$$\mathbb{H}_1 := \{s \in \mathbb{C} : s = \sigma + it, \sigma > 1\}.$$

- To see this, note that

$$|n^{-s}| = |n^{-(\sigma+it)}| = n^{-\sigma}|n^{-it}| = n^{-\sigma}|e^{-i\log(nt)}| = n^{-\sigma}.$$

- Thus,

$$|\zeta(s)| \leq \left|\sum \tfrac{1}{n^s}\right| \leq \sum \left|\tfrac{1}{n^s}\right| = \sum \tfrac{1}{n^\sigma}.$$

- It does not converge if $\sigma \leq 1$.

- But the zeta Function has an analytic continuation to the entire complex plane excluding the point 1.

The product formula for the Riemann's zeta function

Riemann showed that the zeta function can be expressed as a product of its zeros:

$$\zeta(s) = \frac{e^{a+bs}}{s-1} \prod_{\zeta(\alpha)=0} \left(1 - \frac{s}{\alpha}\right) e^{s/\alpha}$$

Now, equating the product for $\zeta(s)$ obtained by Euler with the product of Riemann, taking log and differentiating as we have seen before, Riemann obtains the identity

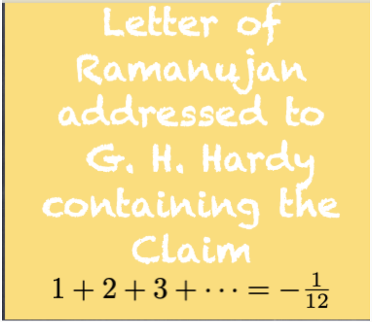
$$\sum_{p^k \leq x} \log(p) = x - \log(2\pi) - \sum_{\zeta(\alpha)=0} \frac{x^\alpha}{\alpha}.$$

The prime number theorem

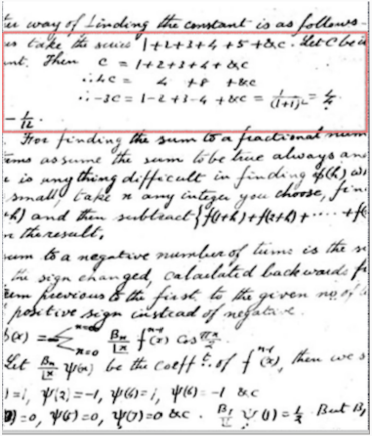
- The Prime Counting Function. The number of primes up to a given quantity x is denoted by $\pi(x)$ (x need not be a whole number).
- The Prime Number Theorem (PNT) states roughly that: $\pi(N)$ behaves very much like $N/\log N$, that is, $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$.
- PNT was conjectured by Gauss at the end of the 18th century, and proved by two mathematicians: Jacques Hadamard and Charles Jean de la Vallée Poussin (independently and simultaneously) at the end of the 19th century, using tools developed by Riemann in the middle of the 19th century.
- If the Riemann Hypothesis is true, it would lead to an exact formulation of PNT, instead of one that is always off by several percent.

Functional equation, analytic continuation

Hardy and Ramanujan

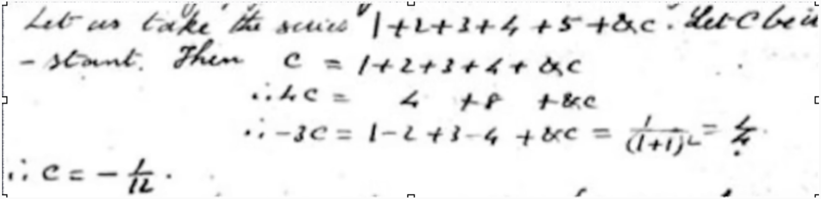


Explaining the intriguing proof of Ramanujan from his notes on the right.



...e way of finding the constant is as follows.
...s take the series 1+2+3+4+5+&c. Let C be the
...nt. Then $C = 1+2+3+4+$ &c
$\therefore 4C = \quad 4 \quad + 8 \quad +$ &c
$\therefore -3C = 1-2+3-4+$ &c $= \frac{1}{(1+1)^2} = \frac{1}{4}$
$-\frac{1}{12}$.
For finding the sum to a fractional num...
...ems assume the sum to be true always an...
... is anything difficult in finding $\phi(h)$ ...
small take n any integer you choose, fin...
...h) and then subtract $\{f(1+h)+f(2+h)+\cdots+f($...
...e the result.
...um to a negative number of times is the s...
the sign changed, calculated backwards f...
...erm previous to the first to the given no of ...
positive sign instead of negative.
$\zeta(x) = \sum_{x=0}^{x=\infty} \frac{B_x}{\lfloor x} f^{x-1}(x) \cos\frac{\pi x}{2}$.
Let $\frac{B_x}{\lfloor x} \psi(x)$ be the coeff^{t}. of $f^{x-1}(x)$, then we s...
...$=1$, $\psi(2)=-1$, $\psi(4)=1$, $\psi(6)=-1$ &c
...$=0$, $\psi(5)=0$, $\psi(7)=0$ &c. $\frac{B_1}{\lfloor 1} \psi(1) = \frac{1}{2}$ But B_1...

Ramanujan’s letter


Let us take the series 1+2+3+4+5+&c. Let C be its con-
-stant. Then $C = 1+2+3+4+\&c$
$\therefore 4C = 4 + 8 + \&c$
$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore C = -\frac{1}{12}$.

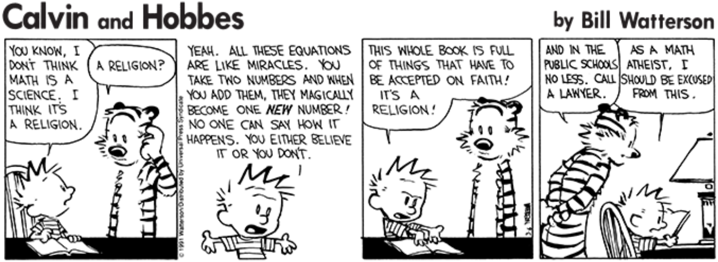
Ramanujan's proof

Let $C = 1 + 2 + 3 + \cdots$. Multiplying C by 4 and subtracting it from C:

$$
\begin{aligned}
C &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\
-4C &= \quad -4 \qquad -8 \qquad -12 \qquad - \cdots \\
-3C &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots
\end{aligned}
$$

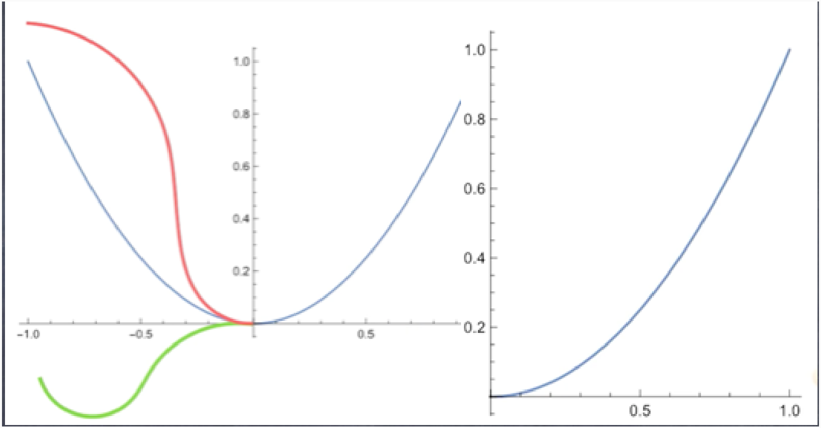
Therefore, $-3C = 1 - 2 + 3 - 4 \cdots \stackrel{?}{=} \frac{1}{(1+1)^2} = \frac{1}{4}$.

Hence $C = -\frac{1}{12}$.



Calvin and Hobbes

Smooth functions



Extending a smooth function

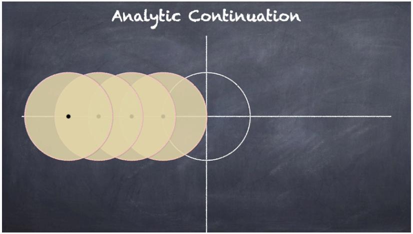
The graph of the smooth function x^2 on the right has several (infinitely many) smooth but distinct extensions.

Complex (analytic) function

Polynomials and convergent power series are examples of analytic functions.

If we are given an analytic function defined on the right half plane, how many different ways, can we extend to the left?

Unlike the case of smooth functions, if there is such an extension, then it is unique.



Analytic functions

Domain stretching

A convergent (infinite) series might define only part of a function; in mathematical terms, it may define a function over only part of that function's domain. The rest of the function might be hidden from our view, waiting to be discovered by some illusive means only a few possess.

Here is a simple example: Let $S(x) = 1 + x + x^2 + x^3 + \cdots$, which converges for $-1 < x < 1$ and equals $1/(1-x)$ for those values of x. Since $1/(1-x)$ makes sense for all numbers except $x = 1$, this shows that the domain of $S(x)$ is larger than $-1 < x < 1$.

The alternating series η

Let us consider the alternating series $\eta(s) := 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \cdots$

Unlike, the Riemann zeta function ζ, the function η is convergent in the entire half plane $\mathbb{H} := \{s \in \mathbb{C} : s = \sigma + it, \sigma > 0\}$.

This follows since the terms of this alternating series decrease to 0 as long as $\sigma > 0$.

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The eta function has a very interesting connection with the zeta function, namely, $\zeta(s) - \eta(s) = \frac{2}{2^s}\zeta(s)$, or equivalently,

$$\zeta(s) = (1 - 2^{1-s})^{-1}\eta(s), \; s \in \mathbb{H}, s \neq 1.$$

This is an instance of analytic continuation of ζ from $\mathbb{H}_1$ to the entire right half plane $\mathbb{H}$ except for 1.

It is clear that if ζ is zero, then η must be also zero, although, η may have more zeros than ζ.

The zeta function and the Gamma function

The integral representation of the Gamma function given below is convergent for $\mathrm{Re}(s) > 0$:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Putting $t = nx$, we have $dt = ndx$ and

$$\Gamma(s) = \int_0^\infty (nx)^{s-1} e^{-nx} ndx = \int_0^\infty n^s x^{s-1} e^{-nx} dx.$$

Thus,

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty x^{s-1} e^{-nx} dx.$$

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Summing over all n, and interchanging the sum and the integral (legitimate when ($\mathrm{Re}(s) > 1$), we have that

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \ \mathrm{Re}(s) > 1.$$

Functional equation for ζ

A slightly different change of variable in $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$, namely, $t = \pi n^2 x \to dt = \pi n^2 dx$ gives the following:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty \left(\pi n^2 x\right)^{\frac{s}{2}-1} e^{-\pi n^2 x}\pi n^2 dx$$
$$= \pi^{\frac{s}{2}} n^s \int^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

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Now, multiplying both sides by n^{-s}, summing over n and then carefully interchanging the sum and the integral, we get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}\Psi(x)dx,$$

where $\Psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Splitting this integral into two parts, and using the symmetry of the function Ψ, one proves that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Zeta values, again!

Here are some interesting values of the zeta function:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

$$\zeta(0) = 1 + 1 + 1 + \ldots = -\frac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + \ldots = -\frac{1}{12}$$

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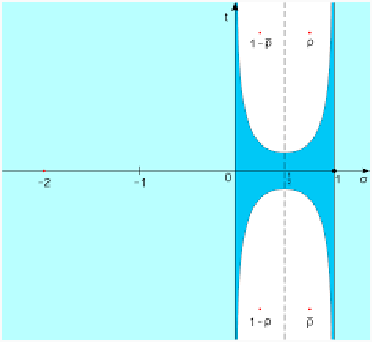
We have already seen that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(1) = \infty$, the harmonic series is divergent. We show that $\zeta(-1) = \frac{-1}{12}$ using the functional equation for zeta and its value at $s = 2$, namely, $\zeta(2) = \frac{\pi^2}{6}$. Thus,

$$\pi^{1/2}\Gamma(-\tfrac{1}{2})\zeta(-1) = \pi^{-1}\Gamma(1)\zeta(2),$$

or equivalently, $\zeta(-1) = -\frac{1}{12}$ since $\Gamma(-\frac{1}{2}) = -2\pi^{1/2}$.

Apart from the trivial zeros, the Riemann zeta function has no zeros to the right of $\sigma = 1$ and to the left of $\sigma = 0$ (neither can the zeros lie too close to those Lines). Furthermore, the nontrivial zeros are symmetric about the real axis and the line $\sigma = \frac{1}{2}$ and, according to the Riemann hypothesis, they all lie on the tine $\sigma = \frac{1}{2}$.



Zeros of the Riemann zeta function

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- zetamath, *The Bessel problem*, https://www.youtube.com/watch?v=FCpRl0NzVu4
- https://www.youtube.com/watch?v=NZYVeO4FHGI&list=PPSV

Thank You!